We want to calculate the $Var[s^2]$ where

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}(x_i^2 - 2x_i\bar{x} + \bar{x}^2) =$$

$$= \frac{1}{n-1}\left(n\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right) - 2n\bar{x}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) + n\bar{x}^2\right) =$$

$$= \frac{n}{n-1}(\bar{x^2} - 2\bar{x}^2 + \bar{x}^2) = \frac{n}{n-1}(\bar{x^2} - \bar{x}^2)$$

now we notice that

$$s^2 = \frac{n}{n-1}(\bar{x^2} - \bar{x}^2) = \frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n}x_i^2 - \frac{n}{n-1}\frac{1}{n^2}\sum_{i=1}^{n}x_i\sum_{j=1}^{n}x_j =$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}x_i^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n}x_i\sum_{j=1}^{n}x_j$$

we can split the second term of the above result in two (from now on if not differently specified the summatories are intended from 1 to $n$):

$$s^2 = \frac{1}{n-1}\sum_i x_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=j}x_ix_j + \sum_{i\neq j}x_ix_j\right)$$

where the last term will vanish when the expectation is taken.

The expected value is then

$$E[s^2] = \frac{1}{n-1}nE[x^2] - \frac{1}{n(n-1)}nE[x^2] = \left(\frac{n}{n-1} - \frac{1}{n-1}\right)E[x^2] \quad (1)$$

Let's now remember that $Var[x] = E[x^2] - E[x]^2$ therefore $Var[s^2] = E[s^4] - E[s^2]^2$, so we only need to find the $E[s^4]$:

$$s^4 = \left[\frac{1}{(n-1)}\sum_i x_i^2 - \frac{1}{n(n-1)}\sum_{i,k}x_ix_k\right]^2 =$$

$$\frac{1}{(n-1)^2}(\sum_i x_i^2)^2 - \frac{2}{n(n-1)^2}(\sum_i x_i^2)(\sum_i x_i)^2 + \frac{1}{n^2(n-1)^2}(\sum_i x_i)^4$$

Let's analyze the single members of the sum:

$$A \equiv (\sum_i x_i^2)^2 = (\sum_i x_i^2)(\sum_j x_j^2) = \sum_i(x_i^2[\sum_{j=i}x_j^2 + \sum_{i\neq j}x_j^2]) =$$

$$= \sum_i x_i^4 + \sum_{i \neq j} x_i^2 x_j^2 = \sum_{i=1}^{n} x_i^4 + 2 \sum_{i<j,1}^{\frac{n(n-1)}{2}} x_i^2 x_j^2$$

Then, at the end we have

$$E[A] = nE[x^4] + n(n-1)E[x^2]^2 \tag{2}$$

We have to do the same with the second member

$$B \equiv (\sum_i x_i^2)(\sum_i x_i)^2 = (\sum_i x_i^2)[\sum_k x_k^2 + \sum_{k \neq j} x_k x_j] =$$

$$= \sum_{i=1}^{n} x_i^4 + \sum_{i \neq j,1}^{2 \cdot \frac{n(n-1)}{2}} (x_i^2 x_j^2 + x_i^3 x_j) + \sum_{i \neq j \neq k} x_i^2 x_j x_k$$

where the terms with odd power members vanishes when the expectation is made so that we have again

$$E[B] = nE[x^4] + n(n-1)E[x^2]^2 \tag{3}$$

Finally the last one:

$$C \equiv (\sum_i x_i)^4 = (\sum_a x_a)(\sum_b x_b)(\sum_c x_c)(\sum_d x_d) =$$

$$= \sum_{i=1}^{n} x_i^4 + \sum_{i \neq j} x_i^3 x_j + \sum_{i<j,1}^{6 \frac{n(n-1)}{2}} x_i^2 x_j^2 + \sum_{i \neq j \neq k} x_i^2 x_j x_k + \sum_{a \neq b \neq c \neq d} x_a x_b x_c x_d$$

now again the second, fourth and last term are zero in the expectation so that we write

$$E[C] = nE[x^4] + 3n(n-1)E[x^2]^2 \tag{4}$$

Finally we have to sum

$$E[s^4] = \frac{1}{(n-1)^2} E[A] - \frac{2}{n(n-1)^2} E[B] + \frac{1}{n^2(n-1)^2} E[C] =$$

$$= \frac{1}{n} E[x^4] + \frac{(n^2-2n+3)}{n(n-1)} E[x^2]^2$$

now we only need to subtract the square of (1) to find

$$Var[s^2] = E[s^4] - E[x^2]^2 = \frac{1}{n} E[x^4] + \frac{(n^2-2n+3)}{n(n-1)} E[x^2]^2 - E[x^2]^2 =$$

$$= \frac{1}{n} \left[ E[x^4] - \frac{n-3}{n-1} E[x^2]^2 \right]$$

Q.E.D.